PARTIAL AVERAGING AND RESONANCE TRAPPING IN A RESTRICTED THREE-BODY SYSTEM

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Abstract.

Based on the value of the orbital eccentricity of a particle and also its proximity to the exact resonant orbit in a three-body system, the Pendulum Approximation (Dermott & Murray 1983) or the Second Fundamental Model of Resonance (Andoyer 1903; Henrard & Lemaître 1983) are commonly used to study the motion of that particle near its resonance state. In this paper, we present the method of partial averaging as an analytical approach to study the dynamical evolution of a body near a resonance. To focus attention on the capabilities of this technique, a restricted, circular and planar three-body system is considered and the dynamics of its outer planet while captured in a resonance with the inner body is studied. It is shown that the first-order partially averaged system resembles a mathematical pendulum whose librational motion can be viewed as a geometrical interpretation of the resonance capture phenomenon. The driving force of this pendulum corresponds to the gravitational attraction of the inner body and its contribution, at different resonant states, is shown to be proportional to e^s, where s is the order of the resonance and e is the orbital eccentricity of the outer planet. As examples of such systems, the cases of (1:1), (1:2) and (1:3) resonances are discussed and the results are compared with known planetary systems such as the Sun-Jupiter-Trojan asteroids.

Key words: celestial mechanics, planetary dynamics, resonance capture, averaging.

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1 INTRODUCTION

The study of the dynamical evolution of a planetary system while captured in a resonance has a long history in dynamical astronomy. Since the pioneering work of Poincaré (1902) on the study of the near-resonance motions in a restricted three-body system by means of a zeroth-order resonance Hamiltonian (i.e., a Hamiltonian with no perturbing terms other than the resonant ones), the body of literature produced on this subject has become so rich and extensive that it is virtually impossible to cite all the articles here. Recent discoveries of extrasolar planetary systems such as Gliese 876 (Marcy et al. 2001), where two planets are locked in a near (2:1) commensurability, have also provided rich grounds for astrodynamicists to extend such studies to the systems beyond the boundaries of our solar system (Laughlin & Chambers 2001; Lissauer & Rivera 2001; Murray, Paskowitz & Holman 2001; Snellgrove, Papaloizou & Nelson 2001; Lee & Peale 2001).

There are two analytical approaches that are commonly taken in the study of the dynamics of the bodies of a three-body system while captured in a resonance. For a particle with high orbital eccentricity (≥ 0.15) in a first-order resonance, or for a particle at any resonance with higher orders, the Pendulum Model is used when the particle's orbit is sufficiently close to the real resonant location (Dermott & Murray 1983). The Hamiltonian Model, or as it is often called, the Second Fundamental Model of Resonance (Andoyer 1903; Henrard & Lemaître 1983) is usually used for small eccentricities. A comparison of the results of the application of these two models to the study of the motion of a test particle in a first-order interior resonance as well as an exterior $(1 : n'), n' = 2, 3, 4, 5$ commensurability, with the results of numerical integrations can be found in a series of papers by Winter & Murray (1997a&b). In these articles, as a part of their comprehensive project CRISS-CROSS on understanding the location and origin of chaotic regions in the phase space of our solar system, Winter and Murray present a detailed analysis of the dynamics of a test particle at resonance.

The purpose of this paper is to present a relatively new approach, namely the method

of partial averaging near a resonance, to study analytically the dynamics of a system near a resonant state. This technique that is based on the Averaging Theorem (see Sanders & Verhulst 1985 and also Wiggins 1996), enables one to avoid certain complexities by studying the behavior of the system averaged over a fast angular variable (Melnikov 1963; Guckenheimer & Holmes 1983; Greenspan & Holmes 1983). It is necessary to mention that a complete picture of the dynamical evolution of the system can only be obtained by direct analysis of its equations of motion. The partially averaged system allows one to focus attention on the slow-changing quantities. Such an idea is commonly used in celestial mechanics: the Hamiltonian of the system is averaged over a fast variable and the resulting averaged Hamiltonian is used to study the slow dynamics of the system. A review of this technique can be found in the work of Ferraz-Mello (1997) and the references cited therein. To demonstrate the capabilities of the method of partial averaging, a restricted, circular and planar three-body system is considered here and the motion of its outer planet near an $(n:n')$ resonance is studied.

Although the method of partial averaging near a resonance has long been used by mathematicians in their studies of dynamical systems near resonances (Melnikov 1963; Guckenheimer & Holmes 1983; Greenspan & Holmes 1983), its application to astronomical systems is quite recent. One can find such applications in papers by this author on study of the dynamical evolution of a planetary system in a uniform and homogeneous disk of planetesimals (Haghighipour 1999, 2000) and also in a series of articles by Chicone, Mashhoon & Retzloff (1996a&b, 1997a&b, 1999, 2000) on their extensive study of the dynamics of a binary system subject to incident gravitational radiation as well as gravitational radiation damping.

The system of interest in this paper is a hypothetical restricted, planar and circular three-body system consisting of a star and two planets. The equations of motion of the outer planet of this system are presented in section 2. Section 3 deals with the system at resonance. In section 4, the method of partial averaging near a resonance is applied and

the averaged dynamics of the outer planet, in the first order of perturbation, is studied. It is shown in this section that the contribution of the gravitational attraction of the inner planet on the averaged dynamics of the outer body at different resonant states is directly proportional to the orbital eccentricity of the latter with a power equal to the order of the resonance. As examples of such cases, the (1:1), (1:2) and (1:3) resonances are studied in detail and a comparison with the system of Sun-Jupiter-Trojan asteroids as an actual case of a near (1:1) commensurability is presented. Section 5 concludes this study by reviewing the results and presenting remarks on their applicability to other planetary systems.

2 THE SYSTEM

The system of interest in this study is a planar three-body system consisting of a star S and two planets P_1 and P_2 where P_2 is the outer planet and P_1, the more massive planet, orbits S uniformly along a circular path. The orbital period of P_1 is considered to be known and constant. It is also assumed that the mass of S is so much larger than P_1 and P_2 that the effect of their gravitational attraction on S can be neglected.

As mentioned earlier, it is the dynamics of P_2 that is of interest here. In an inertial coordinate system with its origin at S and its axes on the plane of the system, the equation of motion of the outer planet can be written as

$$m_2 \frac{d^2 \vec{r}_2}{dt^2} + \mathcal{G} \frac{M m_2}{|\vec{r}_2|^3} \vec{r}_2 + \mathcal{G} \frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} (\vec{r}_2 - \vec{r}_1) = 0 \ . \tag{1}$$

In this equation, $\mathcal{G}$ is the gravitational constant, $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the two planets, m_1 and m_2 are their corresponding masses and M is the mass of the central star. For future purposes, it is more convenient to write equation (1) in a dimensionless form. Introducing r_0 and t_0 as the quantities that carry units of length and time, respectively, equation (1) can be written as

$$\frac{d^2\vec{\hat{r}}}{d\hat{t}^2} + \hat{k}\,\frac{\vec{\hat{r}}}{|\vec{\hat{r}}|^3} + \mu\,\hat{k}\,\frac{(\vec{\hat{r}} - \vec{\hat{r}}_1)}{|\vec{\hat{r}} - \vec{\hat{r}}_1|^3} = 0 \quad , \tag{2}$$

where $\vec{\hat{r}}\,, \vec{\hat{r}}_1$ and $\hat{t}$ are dimensionless quantities given by $\vec{r}_2 = r_0\,\vec{\hat{r}}\,, \vec{r}_1 = r_0\,\vec{\hat{r}}_1$ and $t = t_0\,\hat{t}$. In this equation, $\hat{k} = \mathcal{G}Mt_0^2/r_0^3$ and $\mu = m_1/M$. The assumption of a uniform circular motion for P_1 allows one to set $\hat{k} = 1$ by choosing $r_0 = r_1$ and $t_0 = T_1/2\pi$ where T_1 is the orbital period of P_1.

As mentioned in the previous section, we would like to analyze equation (2) using the method of partial averaging. To do so, it is necessary to write equation (2) in terms of appropriate action-angle variables. The most appropriate action-angle variables for this purpose are the Delaunay variables given by $L = a^{1/2}\,,\, G = \mathcal{P}_\theta = [a(1-e^2)]^{1/2}\,,\, \ell = u - e\sin u$ and $g = \theta - v$ where $e = \mathcal{P}_r\mathcal{P}_\theta/\sin v$ and a are the eccentricity and the semimajor axis of the osculating ellipse of P_2, θ is its plane-polar angle, $\mathcal{P}_r$ and $\mathcal{P}_\theta$ are its radial and angular momenta and v and u are its eccentric and true anomalies related as

$$r = \frac{G^2}{1 + e\cos v} = a\,(1 - e\cos u)\,. \tag{3}$$

In order to write equation (2) in term of the Delaunay variables, it is more convenient to first write this equation in terms of $\mathcal{P}_r$ and $\mathcal{P}_\theta$. That is,

$$\mathcal{P}_r = \dot{r}\,, \tag{4}$$

$$\mathcal{P}_\theta = r^2\,\dot{\theta}\,, \tag{5}$$

$$\dot{\mathcal{P}}_r = \frac{1}{r^3}\,\mathcal{P}_\theta^2 - \frac{1}{r^2} - \frac{\mu}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3}\left[\vec{r} - \cos(\theta - \theta_1)\right], \tag{6}$$

$$\dot{\mathcal{P}}_\theta = -\frac{\mu}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3}\,r\,\sin(\theta - \theta_1)\,, \tag{7}$$

where $\theta_1 = \omega_1\hat{t}$ and $\omega_1 = 1$, is the dimensionless angular velocity of P_1. In equations (4) to (7), the hat signs have been dropped for the sake of simplicity and the overdot indicates

a derivative with respect to the dimensionless time $\hat{t}$. The vector $\vec{\mathbf{r}}_{\mathbf{1}}$ in equations (6) and (7) is the unit vector along $\vec{r}_1$. In terms of the Delaunay variables, equations (4) to (7) can be written as

$$\dot{G} = r\,F_\theta \quad , \tag{8}$$

$$\dot{L} = a\,(1-e^2)^{-1/2}\left[F_\theta + e\,(F_r\,\sin v + F_\theta\,\cos v)\right] \quad , \tag{9}$$

$$\dot{g} = \frac{1}{e}\left[a(1-e^2)\right]^{1/2}\left[F_\theta\Big(\frac{\sin v}{1+e\cos v}\Big) - (F_r\,\cos v - F_\theta\,\sin v)\right] \quad , \tag{10}$$

$$\dot{\ell} = a^{-3/2} + \frac{r}{e}\,a^{-1/2}\left\{(F_r\,\cos v - 2\,F_\theta\,\sin v) + \frac{1}{2}\,e\left[(F_r\,\cos 2v - F_\theta\,\sin 2v) - 3\,F_r\right]\right\}, \tag{11}$$

where

$$F_r = -\,\mu\,\frac{r - \cos(\theta-\theta_1)}{|\vec{r}-\vec{\mathbf{r}}_{\mathbf{1}}|^3} \qquad , \qquad F_\theta = -\,\mu\,\frac{\sin(\theta-\theta_1)}{|\vec{r}-\vec{\mathbf{r}}_{\mathbf{1}}|^3} \quad . \tag{12}$$

3 SYSTEM AT RESONANCE

Consider an $(n : n')$ commensurability between the angular frequency of the inner planet ω_1, and $\omega_\ell = a^{-3/2}$, the Keplerian frequency of the osculating ellipse of the outer one. That is,

$$n\,\omega_1 = n'\,\omega_\ell\;. \tag{13}$$

For our restricted circular system with this resonance condition, the Keplerian frequency ω_ℓ, and therefore, the semimajor axis of the outer planet at resonance are constant. The constant value of the semimajor axis, denoted by $a_{(n:n')}$, corresponds to the resonant value of the action variable L as $L_{(n:n')} = a^{1/2}_{(n:n')}$. We would like to study the dynamics of the outer planet when L varies in the vicinity of this value. For this purpose, we need to introduce an appropriate transformation that renders equations (8) to (11) in a form

that includes deviations of L from $L_{(n:n')}$. Let D be an action variable measuring these deviations. We then write,

$$L = a_{(n:n')}^{1/2} + \mu^{1/2} D, \tag{14}$$

and

$$\ell = a_{(n:n')}^{-3/2} \hat{t} + \varphi, \tag{15}$$

where φ denotes the deviations of the mean anomaly ℓ from its Keplerian value (Appendix A). Such transformations have been repeatedly used in application of Hamiltonian averaging techniques to resonant systems (Ferraz-Mello 1997). It is important to mention that the choice of $\mu^{1/2}$ in equation (14) is to assure equal lowest order of perturbation for $\dot{D}$ and $\dot{\varphi}$ after writing equations (8) to (11) for the system near resonance. Details on this can be found in Wiggins (1996) and also in Haghighipour (2000).

For the purpose of writing equations (8) to (11) near a resonance, it is more convenient to write these equations as (Appendix B)

$$\dot{L} = -\mu \frac{\partial H}{\partial \ell}, \qquad \dot{G} = -\mu \frac{\partial H}{\partial g}, \tag{16}$$

and

$$\dot{\ell} = \omega_\ell + \mu \frac{\partial H}{\partial L}, \qquad \dot{g} = \mu \frac{\partial H}{\partial G}, \tag{17}$$

where $H = -|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^{-1}$ (hereafter, *external Hamiltonian*) represents the perturbative effect of the inner planet, P_1. The dynamical equations of the system near resonance can now be written as (Haghighipour 1999)

$$\dot{D} = -\mu^{1/2} \frac{\partial H}{\partial \ell} - \mu D \frac{\partial^2 H}{\partial \ell \partial L} + O(\mu^{3/2}), \tag{18}$$

$$\dot{\varphi} = -3\mu^{1/2} a_{(n:n')}^{-2} D + \mu \left[6a_{(n:n')}^{-5/2} D^2 + \frac{\partial H}{\partial L} \right] + O(\mu^{3/2}), \tag{19}$$

$$\dot{G} = -\mu \frac{\partial H}{\partial g} + O(\mu^{3/2}), \tag{20}$$

$$\dot{g} = \mu \frac{\partial H}{\partial G} + O(\mu^{3/2}). \tag{21}$$

In these equations, all partial derivatives are evaluated at $(L_{(n:n')}, G, L^{-3}_{(n:n')}\hat{t} + \varphi, g)$. The averaged dynamics of the system is obtained by applying the averaging technique presented in appendix A to equations (18) to (21).

4 FIRST-ORDER AVERAGED SYSTEM

As mentioned before, we would like to study the averaged dynamics of the outer planet near a resonance, in the first order of the perturbation parameter $\mu^{1/2}$. In that order, equations (18) to (21) are written as

$$\dot{D} = -\mu^{1/2}\,\mathcal{F}\,, \qquad \dot{\varphi} = -\mu^{1/2}\left[\frac{3D}{a^2_{(n:n')}}\right], \qquad \dot{G} = \dot{g} = 0\,, \tag{22}$$

where $\mathcal{F} = \partial H/\partial \ell$, is evaluated at $(L_{(n:n')}, G, L^{-3}_{(n:n')}\hat{t} + \varphi, g)$.

In this section, we apply the method of partial averaging, as described in appendix A, to equations (22). As mentioned in that appendix, the analysis presented there is only valid for systems with one angular variable. An inspection of the main dynamical equations of the outer planet (i.e., equations (16) and (17)) reveals that these equations along with $\dot{\theta}_1 = 1$, represent the time variations of two action variables L and G and three angular variables ℓ, g and θ_1. As shown by equations (22), to the first order of $\mu^{1/2}$, $\dot{g} = 0$. Also, from the resonance condition (13) and the transformation (15), the angular variables ℓ and $\theta_1 = \hat{t}$ are related as $\ell = (n/n')\hat{t} + \varphi$. This relation implies that equations (22) represent a dynamical system with an action variable $D(\hat{t})$ and an angular variable $\varphi(\hat{t})$. These equations are now in the correct form for applying the partial averaging technique. Using formula (A7), the averaged dynamics of the outer planet, to the first order of perturbation, can be written as

$$\ddot{\bar{\varphi}} - 3\,\mu\, a^{-2}_{(n:n')}\bar{\mathcal{F}}(L_{(n:n')}\,, G\,, \bar{\varphi}\,, g) = 0\,, \tag{23}$$

where the overbar denotes an averaged quantity.

To study this equation, it only remains to calculate $\mathcal{F}$ which requires one to express H in terms of the mean anomaly ℓ. From its definition, H can be written as

$$H \;=\; -\left[1 \;+\; r^2 \;-\; 2\,r\,\cos(\theta \;-\; \theta_1)\right]^{-1/2} \;. \tag{24}$$

Substituting for r from equation (3) and replacing θ by $g + v$, we have

$$\begin{aligned} H \;=\; -\Bigg\{ &\left[1 + a^2 - 2a\cos(\ell + g - \theta_1)\right] \\ &- e\,a\left[2a\cos\ell + \cos(2\ell + g - \theta_1) - 3\cos(g - \theta_1)\right] \\ &+ \frac{1}{4}\,a\,e^2\Big[2a(3 - \cos 2\ell) + 4\cos(\ell + g - \theta_1) \\ &\qquad - 3\cos(3\ell + g - \theta_1) - \cos(\ell - g + \theta_1)\Big] \;+\; O(e^3)\Bigg\}^{-1/2} , \end{aligned} \tag{25}$$

where $\sin v$ and $\cos v$ have been replaced by

$$\cos v \;=\; -\,e \;+\; 2\left(\frac{1 - e^2}{e}\right)\sum_{j=1}^{\infty}\cos(j\ell)\,J_j(je)\;\;, \tag{26}$$

and

$$\sin v \;=\; (1 - e^2)^{1/2}\sum_{j=1}^{\infty}\sin(j\ell)\,\left[J_{j-1}(je) \;-\; J_{j+1}(je)\right]. \tag{27}$$

Here J_j is the Bessel function of order j. From equation (25), $\mathcal{F}$ can be written as

$$\mathcal{F} \;=\; \mathcal{F}^{(0)} \;+\; e\,\mathcal{F}^{(1)} \;+\; e^2\,\mathcal{F}^{(2)} \;+\; O(e^3)\;\;, \tag{28}$$

where

$$\mathcal{F}^{(0)} \;=\; a\left[1 + a^2 - 2a\cos(\ell + g - \theta_1)\right]^{-3/2}\;\sin(\ell + g - \theta_1)\;\;, \tag{29}$$

$$\begin{aligned} \mathcal{F}^{(1)} \;=&\; a\left[1 + a^2 - 2a\cos(\ell + g - \theta_1)\right]^{-3/2}\left[a\sin\ell + \sin(2\ell + g - \theta_1)\right] \\ &+ \frac{3}{2}\,a^2\left[1 + a^2 - 2a\cos(\ell + g - \theta_1)\right]^{-5/2} \\ &\qquad \sin(\ell + g - \theta_1)\left[2a\cos\ell + \cos(2\ell + g - \theta_1) - 3\cos(g - \theta_1)\right], \end{aligned} \tag{30}$$

and

$$
\begin{aligned}
\mathcal{F}^{(2)} = & \frac{1}{8}\, a \Big[1 + a^2 - 2\,a\,\cos(\ell + g - \theta_1)\Big]^{-3/2} \\
& \qquad \Big[4a\,\sin 2\ell - 4\sin(\ell + g - \theta_1) + 9\sin(3\ell + g - \theta_1) + \sin(\ell - g + \theta_1)\Big] \\
& - \frac{3}{16}\, a^2 \Big[1 + a^2 - 2\,a\,\cos(\ell + g - \theta_1)\Big]^{-5/2} \\
& \qquad \Big[20\,a\,\sin(\ell + g - \theta_1) - 14\,a\,\sin(3\ell + g - \theta_1) \\
& \qquad + 14\,a\,\sin(\ell - g + \theta_1) + 16\sin 2(\ell + g - \theta_1) \\
& \qquad - 7\sin 2(2\ell + g - \theta_1) + 2\,(7 - 4a^2)\,\sin 2\ell - \sin 2(g - \theta_1)\Big] \\
& + \frac{15}{32}\, a^3 \Big[1 + a^2 - 2\,a\,\cos(\ell + g - \theta_1)\Big]^{-7/2} \\
& \qquad \Big[2(4a^2 + 13)\,\sin(\ell + g - \theta_1) + (4a^2 - 7)\,\sin(3\ell + g - \theta_1) \\
& \qquad - (4a^2 - 15)\sin(\ell - g + \theta_1) + \sin(5\ell + 3g - 3\theta_1) \\
& \qquad + 9\,\sin(\ell + 3g - 3\theta_1) + 4a\,\sin 2(2\ell + g - \theta_1) - 16\,a\,\sin 2\ell \\
& \qquad - 12\,a\,\sin 2(g - \theta_1) - 8\,a\,\sin 2(\ell + g - \theta_1) - 6\,\sin 3(\ell + g - \theta_1)\Big]\,.
\end{aligned} \tag{31}
$$

To compute the averaged value of $\bar{\mathcal{F}}$, we expand $[1 + a^2 - 2a\cos(\ell + g - \theta_1)]^{-w}$, $w = 3/2, 5/2, 7/2$, using the identity

$$
(1 - 2\,\xi\,\cos\alpha + \xi^2)^{-\lambda} = \sum_{q=0}^{\infty} C_q^{\lambda}\,(\cos\alpha)\,\xi^q \quad ; \quad |\xi| < 1\,, \tag{32}
$$

taking into account that for the outer planet $a > 1$. In this equation

$$
C_q^{\lambda}(\cos\alpha) = \sum_{h=0}^{q} \frac{\Gamma(\lambda + h)\,\Gamma(\lambda + q - h)}{h!\,(q-h)!\,\Big[\Gamma(\lambda)\Big]^2}\,\cos\big[(q - 2h)\alpha\big] \tag{33}
$$

are the Gegenbauer polynomials. To use identity (32) for the calculation of $\bar{\mathcal{F}}$, one has to set $\alpha = \ell + g - \theta_1$ and $\xi = 1/a$. Simplifying $\mathcal{F}^{(0)}, \mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ using equation (32), one will notice that these quantities will be equal to sum of terms with a general form of

$e^s \cos[(q-2h)(\ell+g-\theta_1)] \sin[\nu\ell+\nu'(g-\theta_1)]$ where ν and ν' are integers. The harmonic nature of these terms requires that in an $(n:n')$ resonance, in order for the formula (A7) to give non-zero values, $(q-2h)$ has to be equal to one of the four integers $\pm(\nu \pm n')$ and, at the same time, one of the four integers $\pm(\nu' \pm n)$ with the same arrangement of + and - signs. This immediately implies that the general term $e^s \cos[(q-2h)(\ell+g-\theta_1)] \sin[\nu\ell+\nu'(g-\theta_1)]$ will have non-zero averaged value only if

$$|\Delta\,\nu| = |\nu - \nu'| = n' - n\,. \tag{34}$$

The quantity $|\Delta\nu|$ in the *selection rule* (34) is, in fact, the order of the resonance. An inspection of equation (28) reveals that in an exterior $(n:n')$ resonance of order $|\Delta\nu|$, the first fulfillment of the condition (34) by the factor $\sin[\nu\ell+\nu'(g-\theta_1)]$ appears where s becomes equal to $|\Delta\nu|$. For instance, the contribution of expansion (28) to the averaged dynamics of the outer planet in a resonance of the form $(n:n+1)$ will first appear in its second term, $\mathcal{F}^{(1)}$, and the third term of this expansion, $\mathcal{F}^{(2)}$, will be the first term with a non-zero averaged value when the system is captured in an $(n:n+2)$ commensurability.

Let us now, just as examples of the first and the second order exterior resonances, study the averaged system near (1:2) and (1:3) commensurabilities. In general, after replacing ℓ by its equivalent value given by equation (15) and averaging the results, the product of $\cos[(q-2h)(\ell+g-\theta_1)]$ and $\sin[\nu\ell+\nu'(g-\theta_1)]$ will produce terms that are proportional to $\sin(n'\bar{\varphi}+ng)$. For instance, for a system near a (1:2) resonance,

$$\bar{\mathcal{F}}_{(1:2)} = \frac{e_{(1:2)}}{a^2_{(1:2)}}\,\sigma_{(1:2)}\,\sin(2\bar{\varphi}+g)\,, \tag{35}$$

and for a (1:3) resonance,

$$\bar{\mathcal{F}}_{(1:3)} = \frac{e^2_{(1:3)}}{4a^2_{(1:3)}}\left[\sigma^{(3/2)}_{(1:3)} + 3\sigma^{(5/2)}_{(1:3)} + \frac{15}{a^2_{(1:3)}}\sigma^{(7/2)}_{(3:1)}\right]\sin(3\bar{\varphi}+g)\,. \tag{36}$$

In these equations,

$$\sigma_{(1:2)} = \sum_{h=0}^{\infty} \left[\frac{\Gamma(\frac{3}{2}+h)}{a_{(1:2)}^{h}\, h!\, \Gamma(\frac{3}{2})}\right]^2 \left\{ 1 + \left(\frac{2h+3}{h+1}\right) \left[1 - \frac{3}{4a_{(1:2)}^2}\left(\frac{2h+5}{h+2}\right)\right]\right\}, \tag{37}$$

$$\sigma_{(1:3)}^{(3/2)} = \sum_{h=0}^{\infty} \left[\frac{\Gamma(\frac{3}{2}+h)}{a_{(1:3)}^{h}\, h!\, \Gamma(\frac{3}{2})}\right]^2 \left\{ \frac{9}{2} + \left(\frac{2h+3}{h+1}\right) \left[1 + \frac{1}{8a_{(1:3)}^2}\left(\frac{2h+5}{h+2}\right)\right]\right\}, \tag{38}$$

$$\begin{aligned} \sigma_{(1:3)}^{(5/2)} = \sum_{h=0}^{\infty} \left[\frac{\Gamma(\frac{5}{2}+h)}{a_{(1:3)}^{h}\, h!\, \Gamma(\frac{5}{2})}\right]^2 &\left\{ \frac{7}{2} + \left[1 - \frac{7}{8a_{(1:3)}^2}\right]\left(\frac{2h+5}{h+1}\right)\right. \\ &\left. - \frac{1}{8a_{(1:3)}^2}\left[7 + \frac{1}{4a_{(1:3)}^2}\left(\frac{2h+9}{h+3}\right)\right]\frac{(2h+7)(2h+5)}{(h+2)(h+1)}\right\}, \end{aligned} \tag{39}$$

$$\begin{aligned} \sigma_{(1:3)}^{(7/2)} = \sum_{h=0}^{\infty} \left[\frac{\Gamma(\frac{7}{2}+h)}{a_{(1:3)}^{h}\, h!\, \Gamma(\frac{7}{2})}\right]^2 &\left\{ \frac{1}{2}\left[a_{(1:3)}^2 - \frac{7}{4}\right] - \frac{3}{4}\left(\frac{2h+7}{h+1}\right)\right. \\ &+ \left[\frac{1+15a_{(1:3)}^2 - 4a_{(1:3)}^4}{32a_{(1:3)}^2}\right]\frac{(2h+9)(2h+7)}{(h+2)(h+1)} \\ &\left. + \frac{3}{16a_{(1:3)}^2}\left[1 - \frac{3}{8a_{(1:3)}^2}\left(\frac{2h+13}{h+4}\right)\right]\frac{(2h+11)(2h+9)(2h+7)}{(h+3)(h+2)(h+1)}\right\}. \end{aligned} \tag{40}$$

Because $\bar{\mathcal{F}}$ is proportional to $\sin(n'\bar{\varphi}+ng)$, equation (23) can be viewed as the equation of a mathematical pendulum with a potential function proportional to $\cos(n'\bar{\varphi}+ng)$ (see equation (A12)). Such a pendulum, with its harmonic potential, is a characteristic of the first-order partially averaged system near a resonance where $e_{(n:n')}$ is considered to be constant (Sanders & Verhulst 1985; Lichtenberg & Lieberman 1992; Wiggins 1996; Haghighipour 1999). For instance, for the cases of (1:2) and (1:3) resonances,

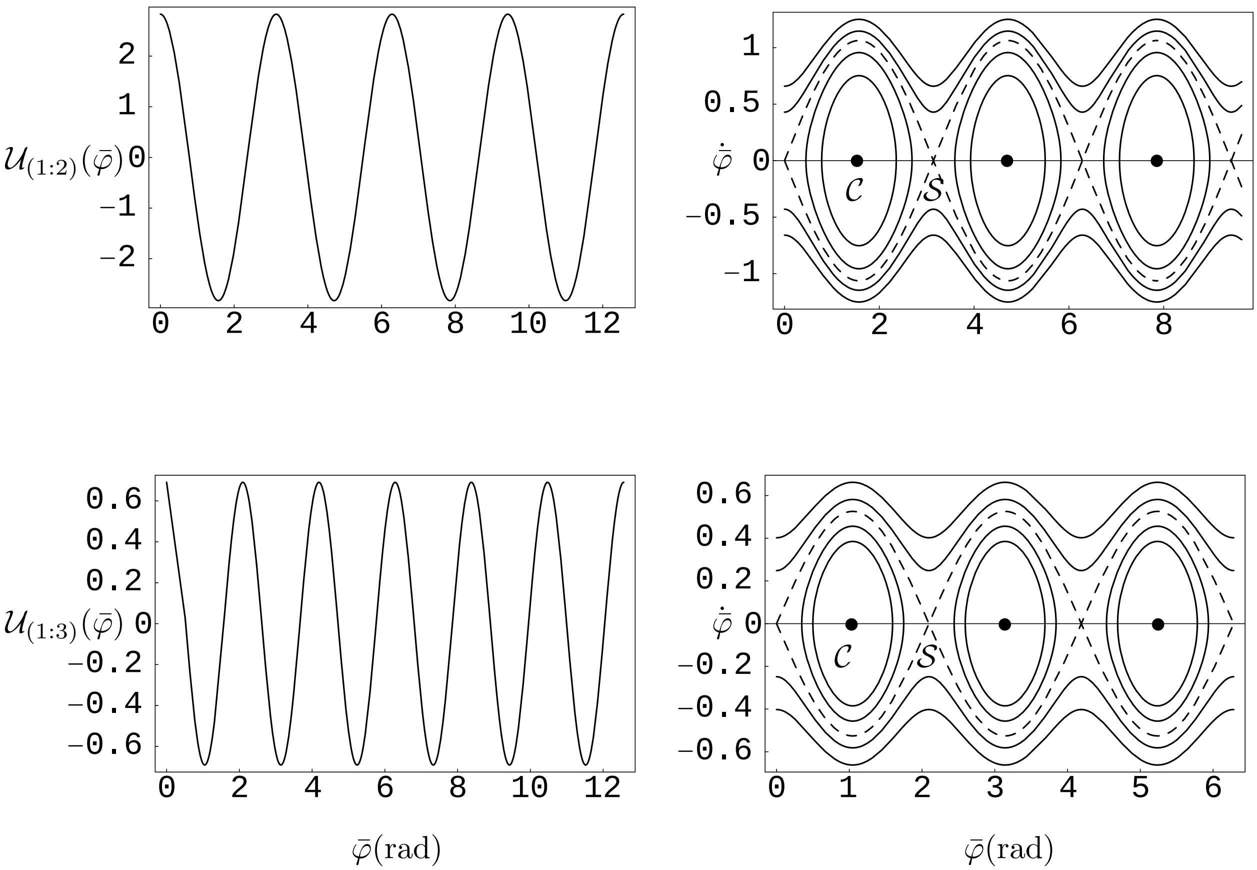

Figure 1. Graphs of the potential function $\mathcal{U}_{(1:2)}(\bar{\varphi})$ (top left) and its phase diagram (top right) and the potential function $\mathcal{U}_{(1:3)}(\bar{\varphi})$ (bottom left) with its associated phase diagram (bottom right) for the system studied by Haghighipour (1999) at (1:2) and (1:3) resonances. The scale on all vertical axes is 0.01 . The origins on the horizontal axes of the graphs of the (1:2) resonance have been shifted by -0.44(rad) and the corresponding origins of the graphs of the (1:3) resonance have been shifted by -1.7(rad).

$$\mathcal{U}_{(1:2)}(\bar{\varphi}) \,=\, \frac{3e_{(1:2)}}{2a^4_{(1:2)}}\,\sigma_{(1:2)}\,\cos(2\bar{\varphi}+g)\,+\,Constant, \tag{41}$$

and

$$\mathcal{U}_{(1:3)}(\bar{\varphi}) \,=\, \frac{e^2_{(1:3)}}{4a^4_{(1:3)}}\Big[\sigma^{(3/2)}_{(1:3)}\,+\,3\,\sigma^{(5/2)}_{(1:3)}\,+\,\frac{15}{a^2_{(1:3)}}\sigma^{(7/2)}_{(1:3)}\Big]\,\cos(3\bar{\varphi}+g)\,+\,Constant. \tag{42}$$

Figure 1 shows the graphs of these two potential functions with their corresponding phase diagrams. In producing these graphs, the numerical values of the orbital eccentricity

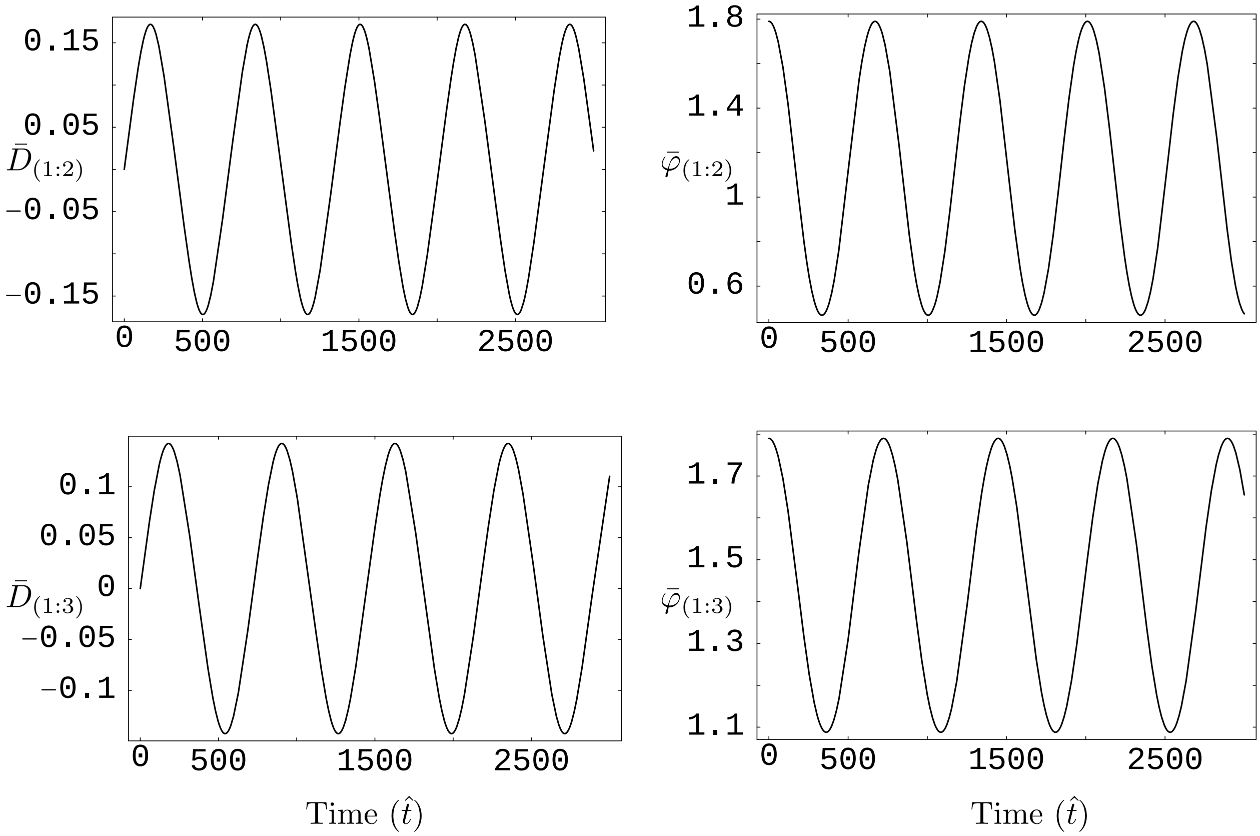

Figure 2. Graphs of $\bar{D}(\hat{t})$ and $\bar{\varphi}(\hat{t})$ for the systems of figure 1, partially averaged to the first order.

and the semimajor axis of P_2 have been taken from the restricted three-body system of Haghighipour (1999) at (1:2) and (1:3) resonances.

The harmonic nature of the potential function of the pendulum (23) indicates that this pendulum can be in three dynamical states; stable equilibrium corresponding to the minimum of the potential function (centers $\mathcal{C}$ on the phase diagrams), unstable equilibrium corresponding to the maximum of the potential function (saddle points $\mathcal{S}$) or, an oscillatory (librational) motion around the stable equilibrium (the orbits inside the *separatrix*, the dashed orbit that passes through the saddle point $\mathcal{S}$). The resonance lock phenomenon is geometrically depicted by these librational motions.

The oscillatory variations in values of $\bar{\varphi}$ create a harmonic behavior for the action variable $\bar{D}$. From equations (22), for a system at an $(n : n')$ resonance where $\bar{\mathcal{F}}$ is pro-

portional to $\sin(n'\bar{\varphi}+ng)$, $\bar{D}$ will be proportional to $\cos(n'\bar{\varphi}+ng)$. Figure 2 shows the graphs of $\bar{D}$ and $\bar{\varphi}$ against time for the systems of figure 1. As mentioned in section 3, D is the measure of changes in the action variable L or, in other words, an indication of the variations of the semimajor axis of the outer planet from its resonant value. From equation (A14), the width of the resonance band within which the semimajor axis of the outer planet varies around its resonant value is limited by the height of the separatrix and to the first order of perturbation can be written as

$$\Delta a_{(n:n')} \;=\; 4\left[\frac{2}{3}\,\mu\, a^3_{(n:n')}\,\Delta\mathcal{U}_{(n:n')}(\bar{\varphi})\right]^{1/2} \tag{43}$$

where $\Delta\mathcal{U}_{(n:n')}(\bar{\varphi})$ is equal to the difference between the maximum and the minimum values of $\mathcal{U}_{(n:n')}(\bar{\varphi})$.

It is necessary to mention that the procedure presented here for expansion of $[1+a^2-2a\cos(\ell+g-\theta_1)]^{-w}$ using Gegenbauer polynomials, is not valid for a (1:1) resonance. At this state, from equation (13), the Keplerian period of the outer planet becomes nearly equal to T_1. That means, $a_{(1:1)} \simeq 1$ and expansion (32) is no longer applicable. However, it is still possible to use the method of averaging to study the dynamics of the outer planet near a (1:1) commensurability. The *external Hamiltonian* H, in this case, must be studied in its entirety as given by equation (25). Expanding H to the second order in eccentricity and integrating $\mathcal{F}$ in the vicinity of the (1:1) resonance, we have

$$\begin{aligned}\bar{\mathcal{F}}_{(1:1)} \;=& \frac{1}{2}\,a_{(1:1)}\,[2\,-\,e^2_{(1:1)}]\,\sin(\bar{\varphi}\,+\,g)\left[1+a^2_{(1:1)}-2a_{(1:1)}\cos(\bar{\varphi}+g)\right]^{-3/2}\\ &-\,3\,a^2_{(1:1)}\,e^2_{(1:1)}\,\sin 2(\bar{\varphi}\,+\,g)\left[1+a^2_{(1:1)}-2a_{(1:1)}\cos(\bar{\varphi}+g)\right]^{-5/2}\\ &+\,\frac{45}{4}\,a^3_{(1:1)}\,e^2_{(1:1)}\sin^3(\bar{\varphi}\,+\,g)\left[1+a^2_{(1:1)}-2a_{(1:1)}\cos(\bar{\varphi}+g)\right]^{-7/2}.\end{aligned} \tag{44}$$

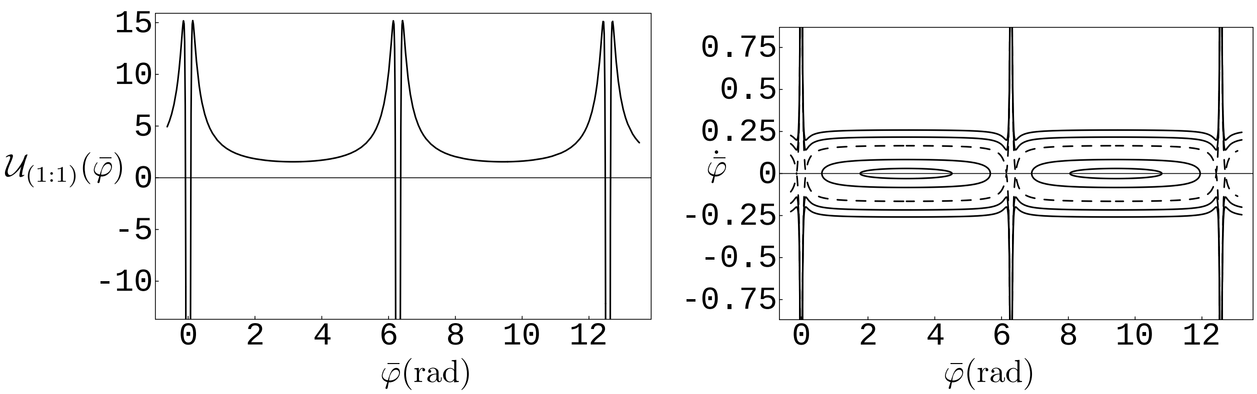

Figure 3 Graphs of the potential function $\mathcal{U}_{(1:1)}(\bar{\varphi})$ (left) and its phase diagram (right) against $\bar{\varphi}$.

The potential function associated with the first-order partially averaged system is given by

$$\begin{aligned}\mathcal{U}_{(1:1)}(\bar{\varphi}) =& \frac{3}{a_{(1:1)}^2}\left[1 + a_{(1:1)}^2 - 2a_{(1:1)}\cos(\bar{\varphi} + g)\right]^{-1/2} \\ &- \frac{21}{2a_{(1:1)}}\, e_{(1:1)}^2\, \cos(\bar{\varphi} + g)\left[1 + a_{(1:1)}^2 - 2a_{(1:1)}\cos(\bar{\varphi} + g)\right]^{-3/2} \\ &+ \frac{27}{4}\, e_{(1:1)}^2 \sin^2(\bar{\varphi} + g)\left[1 + a_{(1:1)}^2 - 2a_{(1:1)}\cos(\bar{\varphi} + g)\right]^{-5/2}.\end{aligned} \tag{45}$$

Figure 3 shows the graph of this potential for $g = 0°$. The fact that $d\mathcal{U}_{(1:1)}(\bar{\varphi})/d\bar{\varphi}$ shows slight deviations from zero near the points of stable equilibrium is an indication of a slow-frequency librational motion in that neighborhood. To see this, let us apply this analysis to the system of Sun-Jupiter-Trojan asteroid. Located at the L_4 and L_5 Lagrangian points of Jupiter's orbit, Trojan asteroids are in a near $(1:1)$ resonance with Jupiter and have a librational motion with a period of approximately 148 years (Brown & Shook 1964; Fleming & Hamilton 2000). Substituting for a and e in equation (45) by the values of the semimajor axis and the eccentricity of Trojan asteroids and Taylor expanding $\mathcal{U}_{(1:1)}(\bar{\varphi})$ around its stable equilibrium, one will obtain a librational period of approximately 110 years. The difference between this period and the 148 years reported in the references above can be attributed to several factors such as neglecting the gravitational effect of

Saturn, restricting Jupiter to stay on a circular orbit and also to the coupling between φ and the argument of the pericenter g. Studies are currently underway to extend the partially averaged equations of this system to the second order of perturbation where φ and g decouple. A closer value for the librational period above is expected in this case.

The first-order partially averaged system near a resonance, presented by equation (23) is, in general, Hamiltonian. It portrays the phenomenon of capture in a resonance as librational motion of a pendulum. However, with respect to an arbitrary perturbation, this Hamiltonian system is structurally unstable. In an actual system, in order to be able to draw conclusions on long-term behavior of quantities such as orbital eccentricity of the outer planet, its angular momentum and also the precession of its orbit, it is necessary to extend this analysis to higher orders of the perturbation parameter $\mu^{1/2}$. Such an extension will allow for the time variation of the angle g to be taken into consideration. This will render the system of equations (18) to (21) in a set of equations with two angular variables, φ and g. In order to be able to apply the method of partial averaging near a resonance to this system, it is then necessary to introduce an averaging transformation that renders the equations of the system in a form that to the first order of perturbation, it becomes automatically equivalent to the first-order partially averaged system at resonance. The second-order partially averaged system is then obtained by averaging those equations using formula (A7). Such studies are currently in preparation for publication.

5. SUMMARY AND CONCLUDING REMARKS

Application of the method of partial averaging to the study of the dynamics of the outer body of a restricted three-body system while captured in a resonance has been presented here. Analysis of the first-order partially averaged system near a resonance has revealed that the equations of motion of the outer planet, averaged over fast periodic motion at resonance, resembles a mathematical pendulum. Such a pendulum analogy can also be found in the comprehensive study of orbital resonances among planetary satellites

by Peale (1986) and also in the comprehensive study of the dynamical behavior of a test particle near an interior as well as an exterior resonance in a restricted three-body system by Winter and Murray (1997a&b).

In the analysis presented here, the driving force of the pendulum-like first-order partially averaged system is obtained from the *external Hamiltonian* H (equation (24)) which involves the gravitational effect of the inner planet on the dynamics of the outer one. In writing the dynamical equations of the outer planet in terms of the Delaunay variables, this force appears as F_r and F_θ in equations (8) to (11). It is important to mention that the form of these equations (i.e., equations (8) to (11)) is quite general and independent of the physical nature of the perturbation. In a system where the perturbations are non-Hamiltonian, extra terms will be added to the functions F_r and F_θ as well as equations (16) and (17). However, equations (8) to (11) will keep their general form (Haghighipour 1999, 2000). The procedure presented here regarding the application of the partial averaging technique and the analysis of the pendulum-like equation are also quite general and can be applied to non-Hamiltonian systems in a similar fashion. In fact, one of the most important features of the method of partial averaging near a resonance is that it presents a general analytical procedure that is equally applicable to both Hamiltonian and non-Hamiltonian systems. As examples of the systems where the method of partial averaging near a resonance has been used in conjunction with non-Hamiltonian perturbations, we refer the reader to Chicone, Mashhoon & Retzloff (1997a&b) and Haghighipour (1999).

As mentioned in section 4, the first-order partially averaged system near a resonance presents the first step in utilizing the method of averaging in the analytical study of the dynamics of a resonance-locked system. To this order of approximation, the argument of the pericenter g was assumed to be constant. In order to obtain a more comprehensive picture of the dynamics of a system near a resonance and the roles that perturbative effects play in its stability, one has to extend this analysis to higher orders of the perturbation parameter $\mu^{1/2}$. Such an extension is necessary to assure decoupling of the angles φ and

g (e.g., see Haghighipour 2000). At that stage, one can apply the averaging technique presented here to the dynamics of the system at the second order of perturbation by averaging those equations over the fast-changing angular variable φ and studying the pendulum-like equation of the angular variable g. Such studies are currently underway for the case of (1:1) resonance and their applications to Trojan asteroids.

The choice of a restricted system as presented in this study was merely to focus attention on the method of partial averaging and its capabilities as another approach to analytical analysis of the dynamics of a system near a resonance. The analysis presented here is quite general and can be applied to any dynamical system at resonance. One can apply such analysis to the systems at interior resonances by setting $\xi = a$ in equation (32) and changing the *selection rule* (34) to

$$|\Delta\nu| = n - n' . \tag{46}$$

An implication of this *selection rule* can be found in a recent paper by Michtchenko & Ferraz-Mello (2001) on analytical modeling of the Jupiter-Saturn system near their (5:2) resonance. They show that at the lowest order, the contribution of the resonant part of the disturbing function appears as the third power of eccentricity, a result that is also implied by the *selection rule* (46).

Other interesting cases for application of the partial averaging technique are the study of the stability of the extrasolar planetary system Gliese 876 (Marcy et al. 2001) where its two planets are locked in a near (2:1) commensurability, the study of dynamical stability of P-type binary planetary systems where recent numerical integrations by Holman & Wiegert (1999) have indicated the existence of islands of instabilities for eccentric binaries at $(n\,:\,1), n > 3$, resonances and also the study of the dynamical evolution of pulsar planetary systems such as PSR B1257 +12 (Konacki, Maciejewski & Wolszczan 1999) and PSR B1620-26 (Thorsett et al. 1999; Ford et al. 2000; Ford, Kozinsky & Rasio 2000; Rasio 2001). It appears that gravitational radiation reaction may play a vital role in the

dynamical evolution of these system (Chicone, Mashhoon & Retzloff 1996 a&b, 1997 a&b, 1999, 2000)

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APPENDIX A: METHOD OF PARTIAL AVERAGING NEAR A RESONANCE

A short introduction to the method of partial averaging near a resonance as used in this article is presented here. For more details on this technique, the reader is referred to Sanders & Verhulst (1985), Wiggins (1996), Haghighipour (1999, 2000) and the references therein.

Partial averaging near a resonance is based on the application of the Method of Averaging to the dynamical equations of a system in the vicinity of its resonant state (Sanders & Verhulst 1985; Wiggins 1996). These equations are usually written in one of the several standard forms (Sanders & Verhulst 1985). In celestial mechanics, it is customary to write the dynamical equations of the system in terms of action-angle variables.

Consider a perturbation system with an action variable $\mathcal{B}$ and an angular variable β such that

$$\dot{\mathcal{B}} \,=\, \varepsilon\,\mathcal{M}(\mathcal{B}\,,\,\beta\,,\,t\,,\,\varepsilon)\,, \tag{A1}$$

and

$$\dot{\beta} \,=\, \omega_0(\mathcal{B}) \,+\, \varepsilon\,\mathcal{Q}(\mathcal{B}\,,\,\beta\,,\,t\,,\,\varepsilon)\,, \tag{A2}$$

where $\mathcal{M}$ and $\mathcal{Q}$ are periodic in time with period $\mathcal{T}$ and ω_0 is the frequency of the unperturbed ($\varepsilon = 0$) system. At resonance $\mathcal{T}$ and ω_0 are related as

$$l\,\omega_0 \,=\, l'\,\omega_{\mathcal{T}}\,, \tag{A3}$$

where $\omega_{\mathcal{T}}$ is the angular frequency associated with $\mathcal{T}$ and l and l' are positive integers. One can show that in the vicinity of the resonance state (A3), equations (A1) and (A2) can be written as (Wiggins 1996; Haghighipour 2000)

$$\dot{\mathcal{E}} \,=\, \varepsilon^{1/2}\,\mathcal{M}(\mathcal{B}_0,\beta,t)\,,+\varepsilon\,\mathcal{E}\,\frac{\partial\mathcal{M}}{\partial\mathcal{B}}(\mathcal{B}_0,\beta,t) \,+\, O(\varepsilon^{3/2}) \tag{A4}$$

and

$$\dot{\Theta} \,=\, \varepsilon^{1/2}\,\mathcal{E}\,\frac{\partial\omega_0}{\partial\mathcal{B}}\,(\mathcal{B}_0) \,+\, \varepsilon\left[\mathcal{Q}(\mathcal{B}_0,\beta,t) \,+\, \frac{1}{2}\,\mathcal{E}^2\,\frac{\partial^2\omega_0}{\partial\mathcal{B}^2}\,(\mathcal{B}_0)\right] \,+\, O(\varepsilon^{3/2})\,, \tag{A5}$$

where

$$\mathcal{E} = \varepsilon^{-1/2}(\mathcal{B} - \mathcal{B}_0) \qquad , \qquad \Theta = \beta - \omega_0(\mathcal{B}_0)t\,, \tag{A6}$$

represent deviations of $\mathcal{B}$ and β from their resonant values $\mathcal{B}_0$ and $\omega_0(\mathcal{B}_0)t$, respectively. Equations (A6) are, indeed, the necessary transformations for writing equations (A1) and (A2) in the neighborhood of the $(l : l')$ resonance. The averaged dynamics of the system at this neighborhood is obtained by averaging equations (A4) and (A5) using the averaging integral

$$\bar{\mathcal{R}} \,=\, \frac{1}{l\mathcal{T}}\int_0^{l\mathcal{T}} \mathcal{R}\big[\mathcal{B}_0\,,\,\omega_0(\mathcal{B}_0)t + \Theta\,,\,t\big]\,dt\,. \tag{A7}$$

The first order partially averaged system near $(l : l')$ resonance is obtained by neglecting the $O(\varepsilon)$ terms in equations (A4) and (A5) and is given by

$$\dot{\bar{\mathcal{E}}} = \varepsilon^{1/2}\, \bar{\mathcal{M}}(\mathcal{B}_0\, ,\, \bar{\Theta})\, , \tag{A8}$$

and

$$\dot{\bar{\Theta}} = \varepsilon^{1/2}\, \bar{\mathcal{E}}\, \frac{\partial \omega_0}{\partial \mathcal{B}}\, (\mathcal{B}_0)\, , \tag{A9}$$

where the overbar indicates an averaged quantity. According to the Principle of Averaging (Saners & Verhulst 1985; Wiggins 1996; Chicone, Mashhoon & Retzloff 1997a; Haghighipour 1999) the dynamics of the system of equations (A1) and (A2) can be approximated by the dynamics of the system (A8) and (A9) during the time interval $\varepsilon^{-1/2}t$. It is necessary to emphasize that in order to be able to make such an approximation, it is required by the Principle of Averaging, that the main dynamical system (i.e., equations (A1) and (A2)) to have only one angular variable. Extension of the method of averaging to the systems with two or more angular variables can be found in the works of Grebenikov & Ryabov (1983), Arnold, Kozlov & Neishtadt (1988) and also in a recent paper by Cucu-Dumitrescu & Selaru (1998) on study of the equations of motion around an oblate planet. In the first order of perturbation, however, such an extension is not necessary.

Introducing $\mathcal{H}$, as

$$\mathcal{H} = \varepsilon^{1/2} \left[\frac{1}{2}\, \bar{\mathcal{E}}^2\, \frac{\partial \omega_0}{\partial \mathcal{B}} (\mathcal{B}_0) \, - \int \bar{\mathcal{M}}(\mathcal{B}_0\, ,\, \bar{\Theta}\, ,\, t)\, d\bar{\Theta} \right] , \tag{A10}$$

one can show that equations (A7) and (A8) can be written as

$$\dot{\bar{\mathcal{E}}} = -\frac{\partial \mathcal{H}}{\partial \bar{\Theta}} \qquad , \qquad \dot{\bar{\Theta}} = \frac{\partial \mathcal{H}}{\partial \bar{\mathcal{E}}}\, . \tag{A11}$$

Equations (A11) imply that $\mathcal{H}$ can be considered as the Hamiltonian of the first-order partially averaged system at resonance. To this Hamiltonian, one can attribute a potential

function given by

$$V(\bar{\Theta}) \,=\, -\int \bar{\mathcal{M}}(\mathcal{B}_0\,,\,\bar{\Theta})\,d\bar{\Theta}\,. \tag{A12}$$

Differentiating equations (A11) with respect to t and using the Hamiltonian $\mathcal{H}$, one can write

$$\ddot{\bar{\Theta}} \,-\, \varepsilon\left[\frac{\partial\omega_0}{\partial\mathcal{B}}(\mathcal{B}_0)\right]\bar{\mathcal{M}}(\mathcal{B}_0,\bar{\Theta}) \,=\, 0\,. \tag{A13}$$

Equation (A13) can be regarded as the equation of a mathematical pendulum with Hamiltonian $\mathcal{H}$ and potential function $V(\bar{\Theta})$. The librational motion of this pendulum presents a geometrical interpretation for the resonance capture phenomenon. The maximum variation of the action variable $\mathcal{B}$ associated with these librational motions is given by (Wiggins 1996)

$$\Delta\mathcal{B} \,=\, 2\left\{2\,\varepsilon\left[\frac{\partial\omega_0}{\partial\mathcal{B}}(\mathcal{B}_0)\right]^{-1}\left[V_{Max}(\bar{\Theta}) \,-\, V_{min}(\bar{\Theta})\right]\right\}^{1/2} \,+\, O(\varepsilon)\,. \tag{A14}$$

APPENDIX B

From definition of H, we have

$$\frac{\partial H}{\partial L} \,=\, \frac{1}{|\vec{r}-\vec{\mathbf{r}}_{\mathbf{1}}|^{-3}}\left\{\left[r-\cos(\theta-\theta_1)\right]\frac{\partial r}{\partial L} \,+\, r\sin(\theta-\theta_1)\,\frac{\partial\theta}{\partial L}\right\}, \tag{B1}$$

$$\frac{\partial H}{\partial G} \,=\, \frac{1}{|\vec{r}-\vec{\mathbf{r}}_{\mathbf{1}}|^{-3}}\left\{\left[r-\cos(\theta-\theta_1)\right]\frac{\partial r}{\partial G} \,+\, r\sin(\theta-\theta_1)\,\frac{\partial\theta}{\partial G}\right\}, \tag{B2}$$

$$\frac{\partial H}{\partial \ell} \,=\, \frac{1}{|\vec{r}-\vec{\mathbf{r}}_{\mathbf{1}}|^{-3}}\left\{\left[r-\cos(\theta-\theta_1)\right]\frac{\partial r}{\partial \ell} \,+\, r\sin(\theta-\theta_1)\,\frac{\partial\theta}{\partial \ell}\right\}, \tag{B3}$$

$$\frac{\partial H}{\partial g} \,=\, \frac{1}{|\vec{r}-\vec{\mathbf{r}}_{\mathbf{1}}|^{-3}}\,r\sin(\theta-\theta_1)\,\frac{\partial\theta}{\partial g}\,. \tag{B4}$$

From these equations it is evident that one needs to compute derivatives of r and θ with respect to all Delaunay variables. From equation (3), we have

$$\frac{\partial r}{\partial L} = -\left(\frac{G}{1+e\cos v}\right)^2 \left(\cos v\,\frac{\partial e}{\partial L} - e\sin v\,\frac{\partial v}{\partial L}\right) \tag{B5}$$

$$\frac{\partial r}{\partial G} = 2\left(\frac{G}{1+e\cos v}\right) - \left(\frac{G}{1+e\cos v}\right)^2 \left(\cos v\,\frac{\partial e}{\partial G} - e\,\sin v\,\frac{\partial v}{\partial G}\right), \tag{B6}$$

$$\frac{\partial r}{\partial \ell} = ea\sin u\,\frac{\partial u}{\partial \ell} \tag{B7}$$

and $\partial r/\partial g = 0$. On the other hand, from $\theta = g + v$, $\partial\theta/\partial g = 1$ and the derivatives of θ with respect to L, G and ℓ will be equal to derivatives of v with respect to these variables. Using $\ell = u - e\sin u$ and $G = L(1-e^2)^{1/2}$ along with equation (3), the partial derivatives of r with respect to the Delaunay variables can be written as

$$\frac{\partial r}{\partial L} = \frac{r}{e}\,a^{-1/2}\left(2e - \cos v - e\cos^3 v\right), \tag{B8}$$

$$\frac{\partial r}{\partial G} = \frac{1}{e}\left[a(1-e^2)\right]^{1/2}\,\cos v\,, \tag{B9}$$

$$\frac{\partial r}{\partial \ell} = e\,a\,(1-e^2)^{-1/2}\,\sin v\,, \tag{B10}$$

and the partial derivatives of θ with respect to L, G and ℓ will be equal to

$$\frac{\partial \theta}{\partial L} = \frac{G^2}{e\,L^3\,(1-e^2)}\,\sin v\,(2 + e\cos v)\,, \tag{B11}$$

$$\frac{\partial \theta}{\partial G} = -\frac{1}{e}\left[a(1-e^2)\right]^{1/2}\left[\frac{1}{r} + \frac{1}{a(1-e^2)}\right]\,\sin v\,, \tag{B12}$$

$$\frac{\partial \theta}{\partial \ell} = \left(\frac{a}{r}\right)^2 (1-e^2)^{1/2}\,. \tag{B13}$$

Replacing the derivatives of r and θ in equations (B1) to (B4) by their equivalent expressions given by equations (B8) to (B13), one can write

$$\frac{\partial H}{\partial L} = \frac{r}{e}\, a^{-1/2} \left\{ \left[\frac{r - \cos(\theta - \theta_1)}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3} \right] \left(2e - \cos v - e\cos^3 v\right) + \left[\frac{\sin(\theta - \theta_1)}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3} \right] (2 + e\cos v)\, \sin v \right\}, \tag{B14}$$

$$\frac{\partial H}{\partial G} = \frac{1}{e} \left[a(1-e^2)\right]^{1/2} \left\{ \left[\frac{r - \cos(\theta - \theta_1)}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3} \right] \cos v - \left[\frac{\sin(\theta - \theta_1)}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3} \right] \left[\frac{1}{r} + \frac{1}{a(1-e^2)} \right] \sin v \right\}, \tag{B15}$$

$$\frac{\partial H}{\partial \ell} = a\,(1-e^2)^{-1/2} \left\{ e \sin v \left[\frac{r - \cos(\theta - \theta_1)}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3} \right] + \frac{a}{r}\,(1-e^2) \left[\frac{\sin(\theta - \theta_1)}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3} \right] \right\}, \tag{B16}$$

and

$$\frac{\partial H}{\partial g} = \frac{r\, \sin(\theta - \theta_1)}{|\vec{r} - \vec{\mathbf{r}}_\mathbf{1}|^3}\,, \tag{B9}$$

which along with equations (12) immediately result in expressions (16) and (17).